L* (Fig. 2.9) by elongating it to infinity and loading its edges with stresses
\[ \sigma_n^+ + i\tau_{ns}^+ = \sigma_n^- + i\tau_{ns}^- = p(t), \quad t \in L, \]  
(2.85)
where upper indexes indicate limit values of respective parameters at contour \( L \) when approaching it from the left (\(+\)) or from the right (\(−\)). It was the last method we applied to solve the above-stated problem.

### 2.4.1.2 Singular Integral Equation of the Problem

The problem for stress distribution in elastic plane with a curvilinear crack will be solved here using the method of singular integral equations [188]. Integral representation of the solution is taken in the form
\[
\Phi(z) = \frac{1}{2\pi} \int_L \frac{g'(t)}{t - z} \, dt,
\]
\[
\Psi(z) = \frac{1}{2\pi} \int_L \left[ \frac{g'(t)}{t - z} \, d\tilde{t} - \frac{\tau(t)}{(t - z)^2} \, dt \right].
\]
(2.86)

Using expressions of stress field components \( \sigma_x, \sigma_y, \tau_{xy} \) in terms of complex potentials \( \Phi(z) \) and \( \Psi(z) \) (1.16) and (1.17), and satisfying the boundary conditions (2.85), we get the singular integral equation of the problem [188]
\[
\frac{1}{\pi} \int_L \left[ K(t, t') \, g'(t) \, dt + L(t, t') \, \frac{g'(t)}{t - z} \, d\tilde{t} \right] = p(t'), \quad t' \in L,
\]
(2.87)
with kernels \( K(t, t') \) and \( L(t, t') \) being given by formulae (1.59).
2.4 Rounded V-Notch Under Symmetrical Loading

A unique solution of integral equation (2.87) in class of functions, which have an integrable singularity at the ends of integration contours exists, if the additional condition is satisfied

$$\int_{L} g'(t) \, dt = 0, \quad (2.88)$$

which ensures uniqueness of displacements during tracing the crack contour.

The equation of crack contour can be written in the parametric form

$$t = l \omega(\xi) = l \begin{cases} (\xi + \xi_0) \cos \beta + \varepsilon \sin \beta + i \left[(\xi + \xi_0) \sin \beta - \varepsilon \cos \beta\right], & -1 \leq \xi < -\xi_0, \\ \varepsilon \cos (\xi/\varepsilon) + i \varepsilon \sin (\xi/\varepsilon), & -\xi_0 \leq \xi \leq \xi_0, \\ - (\xi - \xi_0) \cos \beta + \varepsilon \sin \beta + i \left[(\xi - \xi_0) \sin \beta + \varepsilon \cos \beta\right], & \xi_0 \leq \xi \leq 1, \end{cases} \quad (2.89)$$

where $\xi_0 = 1/(1 + \tilde{\theta})$ is the value of parameter $\xi$ corresponding to straight-to-curvilinear transition point at the crack contour; $\tilde{\theta}$ is straight segments length to circular segment length ratio; $\rho$ is radius of circular segment; $\varepsilon = \rho/l$. The length of crack is $2l = \rho(\pi - 2\beta) \left(1 + \tilde{\theta}\right)$.

As calculations show, the same solution is obtained in the assumption that the crack contour $L$ is infinite and coincides with notch contour $L^*$ ($L = L^*$). Such approach greatly simplifies solution of the problem since there is no need in limit transition between contours. In this approach, equation of notch/crack contour can be written in the parametric form

$$t = \rho \omega(\xi), \quad \omega(\xi) = e^{i\xi\alpha} \begin{cases} -1/ \sin(\xi\alpha - \beta), & -1 < \xi < -\xi_B, \\ 1, & -\xi_B \leq \xi \leq \xi_B, \\ 1/ \sin(\xi\alpha + \beta), & \xi_B < \xi < 1, \end{cases} \quad (2.90)$$

where $\xi_B = (\pi/2 - \beta)/\alpha$ is dimensionless angular coordinate of a contour point $L$, in which the circular arc transforms into the straight segment.

Let us examine in parallel the similar problem for the hyperbolic notch that was solved earlier [15] using the method of Sherman–Lauricella integral equations (see also Sect. 2.2.2). Parametric equation for this problem has the form

$$t = \rho \omega(\xi), \quad \omega(\xi) = e^{-i\xi\alpha} \cos \frac{\alpha}{\cos \alpha - \cos \xi\alpha} + \frac{1}{2} \cot^2 \frac{\alpha}{2} + \frac{1}{2}, \quad \alpha = \pi - \beta, \quad -1 < \xi < 1. \quad (2.91)$$

Vortices of both notches lay in the point $z = \rho$ (Fig. 2.10).

Based on parametric equations (2.90) and (2.91), reduce the Eqs. (2.87) and (2.88) to the dimensionless form
Fig. 2.10 Comparison of two contours $L_1$ (rounded V-notch) and $L_2$ (hyperbolic notch) with identical vertex angles $2\beta = \pi/6$ and notch tip rounding radii $\rho$

\[
\begin{align*}
\frac{1}{\pi} \int_{-1}^{1} \left[ M(\xi, \eta) u(\xi) + N(\xi, \eta) \overline{u(\xi)} \right] \frac{d\xi}{\sqrt{1-\xi^2}} &= p(\eta), \\
\int_{-1}^{1} u(\xi) \frac{d\xi}{\sqrt{1-\xi^2}} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
t' &= \rho \omega(\eta), \\
M(\xi, \eta) &= \rho K(\rho \omega(\xi), \rho \omega(\eta)), \\
N(\xi, \eta) &= \rho L(\rho \omega(\xi), \rho \omega(\eta)), \\
u(\xi) \frac{1}{\sqrt{1-\xi^2}} &= \frac{1}{p} g'(\rho \omega(\xi)) \omega'(\xi), \\
p(\eta) &= \frac{1}{p} p(\rho \omega(\eta)), \quad p = \tilde{K}_1^V/(2\pi \rho)^{1/2}. 
\end{align*}
\]

Make the substitution to increase the accuracy of integral equation (2.92) solution as recommended in [67, 68, 99]

\[
\xi = G(\tau) = a \sinh(\mu \tau), \quad \mu = \text{arsinh} \frac{1}{a}, \quad \eta = G(\xi),
\]

which maps a interval $\tau \in [-1, 1]$ onto the interval $\xi \in [-1, 1]$. Such nonlinear transformation produces thickening of quadrature nodes near the point $\xi = 0$. The constant $a$ in the substitution (2.94) is chosen based on numerical experiments (here we adopted $a = 10^{-5}$).

Now Eq. (2.92) transforms into the following

\[
\begin{align*}
\frac{1}{\pi} \int_{-1}^{1} \left[ M(\xi, \eta) u^*(\tau) + N(\xi, \eta) \overline{u^*(\tau)} \right] \frac{d\tau}{\sqrt{1-\tau^2}} &= p(\eta), \\
\int_{-1}^{1} u^*(\tau) \frac{d\tau}{\sqrt{1-\tau^2}} &= 0,
\end{align*}
\]

\[(2.95)\]
where the following designation was introduced

\[ \frac{u^*(\tau)}{\sqrt{1 - \tau^2}} = \frac{u(\xi) G'(\tau)}{\sqrt{1 - \xi^2}}. \]  

(2.96)

Solve the Eq. (2.95) numerically using quadrature method and applying Gauss–Chebyshev quadratures (1.100) and (1.101) to compute integrals. The system of linear complex algebraic equations will result

\[
\begin{aligned}
\frac{1}{2n} \sum_{k=1}^{2n} [M(\xi_k, \eta_m) u^*(\tau_k) + N(\xi_k, \eta_m) u^*(\tau_k)] &= p(\eta_m), \\
\frac{1}{2n} \sum_{k=1}^{2n} u^*(\tau_k) &= 0,
\end{aligned}
\]

\[ m = 1, \ldots, (2n - 1), \]  

(2.97)

where

\[ \xi_k = G(\tau_k), \quad \tau_k = \cos \frac{\pi (2k - 1)}{4n}, \quad k = 1, \ldots, 2n; \]  \[ \eta_m = G(\zeta_m), \quad \zeta_m = \cos \frac{\pi m}{2n}, \quad m = 1, \ldots, 2n - 1. \]  

(2.98)

(2.99)

The problem is symmetrical with respect to axis \( x \), thereby providing satisfaction of symmetry condition [208]

\[ u^*(-\tau) = u^*(\tau). \]  

(2.100)

Taking into account relationship (2.100), one can twice reduce order of the system (2.97). As a result, we come to the following system of algebraic equations:

\[
\begin{aligned}
\frac{1}{2n} \sum_{k=1}^{n} [M^*(\xi_k, \eta_m) u^*(\tau_k) + N^*(\xi_k, \eta_m) u^*(\tau_k)] &= p(\eta_m), \\
\frac{1}{2n} \sum_{k=1}^{2n} [u^*(\tau_k) + \overline{u^*(\tau_k)}] &= 0,
\end{aligned}
\]

\[ m = 1, \ldots, n, \]  

(2.101)

where

\[
\begin{aligned}
M^*(\xi_k, \eta_m) &= M(\xi_k, \eta_m) + N(-\xi_k, \eta_m), \\
N^*(\xi_k, \eta_m) &= N(\xi_k, \eta_m) + M(-\xi_k, \eta_m).
\end{aligned}
\]  

(2.102)

For the collocation node \( \eta_n = 0 \) at the axis of symmetry (\( x \) axis), respective complex equation of the system (2.101) is reduced to real one due to symmetry
of the problem. Finally, at \( m = 1, \ldots, n \), we get \( 2n - 1 \) real equations, which create, together with the last real equation, a closed system of \( 2n \) real algebraic equations for \( n \) complex unknown functions \( u^*(\tau_k) \), \( k = 1, \ldots, n \).

### 2.4.2 Symmetrical Stress Distribution in Plane with Rounded V-Notch [192]

#### 2.4.2.1 Stresses at Notch Contour

Tangential normal stresses at right edge of crack/notch \( L \) (at contour of rounded V-notch) are derivable from the relationship

\[
\sigma^*_s = 4\text{Re} \left[ \Phi^+_0(t) + \Phi^-(t) \right], \quad t \in L, \quad (2.103)
\]

which follows from (1.159) in absence of loads at notch contour.

Boundary value of potential \( \Phi(z) \) at contour \( L \) is computable using Sokhotski–Plemelj formula (1.34). Considering substitutions (2.93) and (2.94) as well as first of formulae (2.65), one gets

\[
\sigma^*_s(\eta) = \frac{4}{(2\pi \rho)^{\lambda_1}} \text{Re} \left[ \frac{-\sin 2\alpha}{(\lambda_1 - 1) \sin 2\alpha + \sin 2\lambda_1 \alpha} \frac{1}{[\omega(\eta)]^{\lambda_1}} + \right.
\]

\[
- \frac{i}{2} \frac{u^*(\xi)}{G'(\xi)\omega'(\eta)\sqrt{1 - \xi^2}} + \frac{1}{2\pi} \int_{-1}^{1} \frac{u^*(\tau)}{\omega(\xi) - \omega(\eta)} \frac{d\tau}{\sqrt{1 - \tau^2}} \right]
\]

\[
= \frac{\tilde{K}_1^V}{(2\pi \rho)^{\lambda_1}} R_1(\beta, \eta). \quad (2.104)
\]

Apply the quadrature formula (1.100) to compute singular integral in relationship (2.104). Using the condition of symmetry (2.100), we can find dimensionless stress \( R_1(\beta, \eta_j) \) in nodes \( \zeta_j = \cos(j\pi/(2n)), j = 1, \ldots, (2n - 1) \) from the formula

\[
R_1(\beta, \eta_j) = 4\text{Re} \left\{ \frac{-\sin 2\alpha}{(\lambda_1 - 1) \sin 2\alpha + \sin 2\lambda_1 \alpha} \frac{1}{[\omega(\eta_j)]^{\lambda_1}} + \right.\]

\[
- \frac{i}{2} \frac{u^*(\zeta_j)}{G'(\zeta_j)\omega'(\eta_j)\sqrt{1 - \zeta_j^2}} + \frac{1}{4n} \sum_{k=1}^{n} \left[ \frac{u^*(\tau_k)}{\omega(\xi_k) - \omega(\eta_j)} + \frac{u^*(\tau_k)}{\omega(\xi_k) - \omega(\eta_j)} \right] \right\}. \quad (2.105)
\]
2.4 Rounded V-Notch Under Symmetrical Loading

Fig. 2.11 Comparison of dimensionless stress distributions along rounded V-notch (solid curve) or hyperbolic notch (dashed curve) for various vertex angles at symmetrical loading.

Values of function $u^*(\tau)$ in arbitrary point $\tau \neq \tau_k$ are computable with the help of interpolation formula (1.98). Again using the condition of symmetry (2.100), we get the following relation

$$u(\tau) = \frac{1}{2n} \sum_{k=1}^{n} (-1)^k T_{2n}(\tau) \sqrt{1 - \tau_k^2} \left[ \frac{u(\tau_k)}{\tau + \tau_k} - \frac{u(\tau_k)}{\tau - \tau_k} \right]. \quad (2.106)$$

Since in nodes $\zeta_m = \cos(\pi m/(2n))$ ($m = 1, \ldots, n - 1$) Chebyshev polynomial is $T_{2n}(\zeta_m) = (-1)^m$, we have

$$u(\zeta_m) = \frac{1}{2n} \sum_{k=1}^{n} (-1)^{k+m} \sqrt{1 - \tau_k^2} \left[ \frac{u(\tau_k)}{\zeta_m + \tau_k} - \frac{u(\tau_k)}{\zeta_m - \tau_k} \right]. \quad (2.107)$$

Authors [111, 112] had calculated dimensionless stress $R_l(\beta, \xi)$ along notch contour for vertex angles $2\beta \in [0, \pi]$ and two different contour geometries $L$ (2.90) (rounded V-notch) or (2.91) (hyperbolic notch) with identical radii of curvature in tips $\rho$ (Fig. 2.11). We can see that stress distributions along notch contour is essentially different in these two cases. Their relative differences reach 10% in the notch tip.

2.4.2.2 V-Notch Stress Rounding Factor

In the notch tip $z = \rho$ ($\xi = 0$) dimensionless stress $R_l(\beta, 0)$ reaches the maximal value $R_l(\beta) = R_l$. The effect of notch tip rounding on maximal stresses is described by expression [15]

$$\left(\sigma_s^*\right)_{\text{max}} = \frac{\tilde{K}_V}{(2\pi)^{\lambda_1}} R_l \rho^{-\lambda_1}. \quad (2.108)$$
The stress rounding factor in the expression is computable using formula (2.105). We can find values of functions $u^*(\tau)$ in the point $\tau = 0$ using the interpolation polynomial (2.107), which for this case has a simplified form

$$u^*(0) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+n} \text{Re} \ u^*(\tau_k) \tan \frac{\pi(2k-1)}{4n}. \quad (2.109)$$

Calculations of stress rounding factor values $R_I(\beta)$ were performed for notch vertex angles $2\beta$ changing in the interval $0 \leq 2\beta < \pi$ with increment $\pi/360$. So fine resolution was necessary to construct a sufficiently accurate fitting function $R_I(\beta)$ for rounded V-notch. We had estimated the accuracy of stress rounding factor $R_I$ determination procedure in the following way: the order of algebraic equations systems (2.101) was doubled until the relative difference $R_I$ for the given angle $\beta$ became less than 0.1%.

For hyperbolic notch (2.91), obtained values of factor $R_I$ were in good accordance with known results [15], the relative difference being below 0.1%, which confirmed correctness of present calculations. Figure 2.12 demonstrates dependence of the factor $R_I(\beta)$ on vertex angle $2\beta$. At $2\beta = \pi$ both curves gain the obvious value $R_I = 1$. At $\beta = 0$, when the hyperbolic notch transforms into the parabolic one, the

---

**Fig. 2.12** Influence of notch geometry on factor $R_I$ at various vertex angles $2\beta$: rounded V-notch (1) versus hyperbolic notch (2)

**Fig. 2.13** U-notch in elastic plane
value $R_I = 2\sqrt{2}$ is observed. At $2\beta = 0$, when the rounded V-notch transforms into a U-shaped notch (Fig. 2.13), observed value $R_I = 2.992$ is close to known results [128, 142]. Relative differences of results for hyperbolic and rounded V-notches reaches $6 \div 10\%$ at vertex angles $2\beta < 2\pi/3$ that confirms the essential influence of notch geometry on maximal stresses near notch tip.

Note that maximal stresses (2.108) in the tip of U-shaped notch ($\lambda_I = 1/2$) virtually coincide with respective values for semi-infinite crack with a circular hole of the radius $\rho$ in its tip [47]

$$\sigma_{\text{max}} = 2.991 \frac{K_I}{\sqrt{2\pi\rho}}, \quad (2.110)$$

where $K_I$ is stress intensity factor at a crack tip. An asymptotic solution had been built for the last problem as well [129].

Since the stress rounding factor $R_I$ for rounded V-notch will be often used in next sections, calculated values of $R_I$ are presented in Table 2.6 for individual notch vertex angles.

These results were used to construct the fitting expression [203]

$$R_I = \frac{1 + 28.75\gamma + 98.04\gamma^2 - 102.1\gamma^3 + 47.42\gamma^4 - 8.441\gamma^5}{1 + 20.71\gamma}, \quad \gamma = \pi/2 - \beta, \quad (2.111)$$

which provides relative errors less than 0.1% in the interval $\beta \in [0^\circ, 83^\circ]$ and 0.4% in the interval $\beta \in [83^\circ, 90^\circ]$. Coefficients at $\gamma^4$ and $\gamma^5$ in the formula (2.111) slightly differ from those published in [192, 193], that gave us a possibility to reach higher accuracy.

Above-presented analysis shows that interrelation between stress intensity and stress concentration factors for sharp and rounded notches depends not only on radius of curvature in the notch tip, but also on notch shape near the tip. In past, many researchers had believed that the relation (2.44) is precise for narrow U-shaped notches ($\beta = 0$), that is the difference between parabolic and U-shaped notches was neglected (see, e.g., [88, 90, 135]). They had paid attention to only radius of curvature in notch tip and ignored the notch shape in some vicinity of its tip.

**Table 2.6** Values of stress rounding factor $R_I(\beta)$ for rounded V-notch

<table>
<thead>
<tr>
<th>$2\beta$</th>
<th>$0^\circ$</th>
<th>$1^\circ$</th>
<th>$5^\circ$</th>
<th>$10^\circ$</th>
<th>$15^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_I$</td>
<td>2.992</td>
<td>2.992</td>
<td>2.993</td>
<td>2.994</td>
<td>2.995</td>
<td>2.999</td>
<td>2.997</td>
<td>2.986</td>
</tr>
<tr>
<td>$2\beta$</td>
<td>75°</td>
<td>90°</td>
<td>105°</td>
<td>120°</td>
<td>135°</td>
<td>150°</td>
<td>165°</td>
<td>180°</td>
</tr>
<tr>
<td>$R_I$</td>
<td>2.957</td>
<td>2.901</td>
<td>2.806</td>
<td>2.659</td>
<td>2.439</td>
<td>2.123</td>
<td>1.677</td>
<td>1.000</td>
</tr>
</tbody>
</table>


2.5 Rounded V-Notch Under Mixed Loading

2.5.1 Antisymmetric Stress Distribution \[ 204 \]

2.5.1.1 Problem Statement

Let the elastic plane contain sharp V-notch with the tip in coordinate system origin and vertex angle \( 2\beta \) \((0 \leq 2\beta < \pi)\) (Fig. 2.8a). Assume that stress state of the notched plane is determined by complex potentials \( \Phi_0^a(z) \) and \( \Psi_0^a(z) \) (2.70), which ensure zero stresses at the notch contour \( L^0 \). Let us consider the smooth contour \( L \) composing of straight segments parallel to wedge faces \( L_0 \) and circular arc with radius \( \rho \) and center in the notch tip. Let us write the vector of normal \((\sigma_n^0)\) and shear \((\tau_{ns}^0)\) stresses at this contour

\[
\sigma_n^0 + i\tau_{ns}^0 = \Phi_0^a(t) + \Phi_0^a(t) + \frac{d}{dt} \left[ t\Phi_0^a(t) + \Psi_0^a(t) \right] = -p(t), \quad t \in L^*.
\]

Now consider the rounded V-notch with the same vertex angle and free of stresses contour \( L^* \) in the plane (Fig. 2.8b). Let an asymptotic stress distribution is given at infinity, which is determined by potentials

\[
\Phi_0^a(z) = \frac{i\tilde{K}_\Pi^V}{(2\pi z^\alpha)} \sin 2\alpha, \\
\Psi_0^a(z) = \frac{i\tilde{K}_\Pi^V}{(2\pi z^\alpha)} \sin 2\alpha.
\]

We shall apply a superposition technique to solve this boundary value problem. Write above stress potentials in the form

\[
\Phi_s(z) = \Phi_0^a(z) + \Phi(z), \quad \Psi_s(z) = \Psi_0^a(z) + \Psi(z),
\]

where functions \( \Phi(z) \) and \( \Psi(z) \) describe the disturbed stress state induced by a rounded V-notch \( L^* \).

To find the disturbed stress state, we have to solve the boundary value problem for elastic plane containing the rounded V-notch with a contour \( L^* \), at which the boundary condition

\[
\sigma_n + i\tau_{ns} = p(t), \quad t \in L^*,
\]

is satisfied, and stresses vanish at infinity. Stresses \( p(t) \) here are determined from the formula (2.112).
2.5.1.2 Singular Integral Equation

Above-stated boundary value problem will be solved here using the method of singular integral equations similarly to previous case of symmetrical loading. Namely, we shall reduce it to boundary value problem for crack/notch along contour $L$, which, in limit case when the crack length approaches infinity, approaches the contour $L^*$ (2.90). Integral representation of the solution is taken in the form (2.86). Satisfying the boundary conditions at crack edges, we get the singular integral equation of the problem

$$\frac{1}{\pi} \int_L \left[ K(t, t')g'(t) dt + L(t, t')\overline{g'}(t') d\overline{t} \right] = p(t'), \quad t' \in L,$$

(2.116)

with kernels being given by the formulae (1.59).

A unique solution of integral equation (2.116) in class of functions, which have an integrable singularity at the ends of integration contours, exists if the additional condition is satisfied during tracing the crack contour

$$\int_L g'(t) dt = 0.$$

(2.117)

Solve the Eq. (2.116) numerically under condition (2.117) and get results for rounded V-notch (2.90) and hyperbolic notch (2.91). Write the Eqs. (2.116) and (2.117) in the canonical dimensionless form

$$\begin{align*}
\frac{1}{\pi} \int_{-1}^{1} & \left[ M(\xi, \eta)\phi(\xi) + N(\xi, \eta)\overline{\phi}(\xi) \right] d\xi = p(\eta), \quad -1 \leq \eta \leq 1, \\
\frac{1}{\pi} \int_{-1}^{1} & \phi(\xi)\omega'(\xi) d\xi = 0,
\end{align*}$$

(2.118)

where

$$\begin{align*}
\phi(\xi) &= \frac{1}{\tau} g'(\rho\omega(\xi)), \\
M(\xi, \eta) &= \rho\omega'(\xi) K(\rho\omega(\xi), \rho), \\
N(\xi, \eta) &= \rho\omega'(\xi) L(\rho\omega(\xi), \rho\omega(\eta)), \\
p(\eta) &= \frac{1}{\tau}p(\rho\omega(\eta)), \quad \tau = \frac{\tilde{K}_V}{(2\pi \rho) \lambda_{II}}.
\end{align*}$$

(2.119)

Since we are considering the infinite contour $L$ and in its ends (that is in points $\xi \pm 1$) unknown function $\phi(\xi)$ is constrained, we shall seek this function in the class